Mathematical Concepts for Computer Science *Random Variables.*

Question 1. Let $X_1, X_2, \ldots, X_n$ be independent Exponential random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively (also referred to as *rates*).

(i) Let $Y = \min_i \mathrm{X}_i$. Calculate $\mathbb{P}(Y \geq y)$ as a function of y. Show that $Y \sim \mathrm{Exp}(\lambda_1 + \cdots + \lambda_n)$.

(ii) It can be shown that if A and B are two independent exonential random variables with rates a and b, then $\mathbb{P}(A \leq B) = a/(a+b)$. Use this fact to show

$$\mathbb{P}(Y = X_1) = \frac{\lambda_1}{\lambda_1 + \cdots + \lambda_n}.$$

Deduce that

$$\mathbb{P}(Y = X_j) = \frac{\lambda_j}{\lambda_1 + \cdots + \lambda_n}.$$

(iii) Let $Z = \max_i \mathrm{X}_i$. Find the distribution of Z, i.e. calculate $\mathbb{P}(Z \geq z)$ as a function of z.
Hint. First find $\mathbb{P}(Z < z)$.

Question 2.

(i) Let $X \sim$ Exponential(λ). Calculate $\mathbb{P}(X \geq x + x_o | X \geq x_o)$. Show that this is equal to $\mathbb{P}(X \geq x)$. This is called the *memoryless property* of the exponential distribution.

(ii) You observe bus inter-arrival times of 3 minutes, 7 minutes, 10 minutes, 2 minutes and 8 minutes. You suspect that these are independent Exponential(λ) random variables. Estimate λ.

(iii) Let $X \sim$ Pareto(α). Calculate $\mathbb{P}(X \geq x + x_o | X \geq x_o)$. How does this compare to $\mathbb{P}(X \geq x)$?

(iv) Let $X \sim$ Normal$(0, 1)$. Investigate $\mathbb{P}(X \geq x + x_o | X \geq x_o)$ numerically.

Question 3.
For the Uniform[a,b] distribution find the mean, median, mode and variance.

Question 4.
Suppose that $X_1, \ldots, X_n$ are independent, identically distributed random variables with mean μ and variance σ^2. Find the mean of

$$S^2 = \sum_{1=1}^{n} \left(X_i - \overline{X}\right)^2, \text{ where } \overline{X} = \frac{1}{n} \sum_{1}^{n} X_i$$

Question 5.
For a random variable X with mean μ and variance σ^2 define the function $V(x) = \mathbb{E}(X - x)^2$. Express the random variable $V(X)$ in terms of μ, σ^2 and X, and hence show that $\sigma^2 = \frac{1}{2}\mathbb{E}(V(X))$.