Induction Problems

September 4, 2009

1. The king summoned the best mathematicians in the kingdom to the palace to find out how smart they were. The king told them "I have placed white hats on some of you and black hats on the others. You may look at , but not talk to one another. I will leave now and will come back every hour on the hour. Every time I return, I want those of who have determined that you are wearing white hats to come up and tell me immediately." As it turned out, at the nth hour every one of the 'n' mathematicians who were given white hats informed the king that she knew that she was wearing a white hat? Why?

2. Let Σ be the alphabet $\{a, b, c\}$. Show that the number of words of length 'n' in which the letter 'a' appears an even number of times is $(3^n + 1)/2$. (Use Induction or any other proof technique)

3. Prove that the total number of nodes in a full binary tree of height h is $2^{h+1} - 1$.

4. Prove that the minimum number of nodes in an AVL tree of height 'h' is $F_{h+2} - 1$ where 'F' is a Fibonacci number with $F_0 = 1$, $F_1 = 1$, $F_2 = 2$ .... .

5. Prove by induction on 'n' that the number of subsets of size 'k' of a set of size 'n' is given by the expresssion : $\frac{n!}{k!(n-k)!}$

6. Show that the number of binary relations on a set of size 'n' is 2^{n^2}.

7. In how many different ways can a boy climb up a flight of n steps, if he can climb skip atmost 2 steps at a time. Prove your answer using induction.

8. Prove:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} .... + \frac{1}{2^n} < 1 \tag{1}$$

9. Prove by induction that a number, given in its decimal representation, is divisible by 3 if and only if the sum of its digits is divisible by 3.

10. Prove by induction that

$$(1 + 2 + 3 + ... + n)^2 = 1^3 + 2^3 + 3^3 + ... + n^3 \tag{2}$$

11. Show that the number of distinct triangles that can be formed by using the vertices of a convex regular polygon with n ($n > 3$) vertices is $\frac{n(n-1)(n-2)}{6}$. Prove the induction on n.