Example sheet 3

Random processes

Network Performance—DJW—2008/9

Question 1. Draw the state space diagrams for the following transition matrices. Is the corresponding Markov chain irreducible? aperiodic? If both irreducible and aperiodic, calculate the equilibrium distribution.

$$\text{(i). } P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\text{(ii). } P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \varepsilon & 1-\varepsilon \\ 1 & 0 & 0 \end{pmatrix} \text{ where } 0 < \varepsilon < 1$$

$$\text{(iii). } P = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Question 2. Explain carefully why the following Markov chain transition matrix is aperiodic.

$$\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Question 3. I have a biased coin, which has probability p of heads. I toss it repeatedly. I stop when I first get three heads in a row. Let X_n be the result of the nth toss. Draw a state space diagram for the triple (X_n, X_{n-1}, X_{n-2}). Hence or otherwise, calculate the expected time until I stop.

Question 4. My website consists of 5 pages, 1–5. The link matrix

$$L = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

shows which pages link to which other pages: $L_{ij} = 1$ if i links to j, and 0 otherwise. There are two outside nodes A and B which link to my website: A links to my page 1, and B links to my page 3. A has PageRank= 2 and 3 other outgoing links; B has PageRank= 5 and 2 other outgoing links.

Add a fictitious node X which represents the 'rest of the web'. Consider a random surfer who follows the Google PageRank Markov chain, and suppose that the random teleports all go to X, and that the extra outgoing links from A and B also go to X. Suppose that, from X, the random surfer goes to A with probability $2/7$ and to B with probability $5/7$.

(i) Draw the state space diagram for this Markov chain.
(ii) Calculate the equilibrium distribution π. What is the total fraction of time that this Markov chain spends in my website, i.e. in states 1–5?
(iii) Calculate the expected length of a visit to my website, assuming the first page is 1. Calculate the expected length of a visit to my website, assuming the first page is 3. Hence calculate the expected length of a visit to my website, assuming that the first page is 1 with probability $\pi_1/(\pi_1 + \pi_3)$ and 3 otherwise.
(iv) Consider adding a navigation bar, i.e. links from $2 \to 1$, $4 \to 2$ and $4 \to 1$ etc. What is π now? What is the expected length of a visit to my website now?
(v) Comment on how reasonable this Markov chain approach is for improving my PageRank.

Question 5. Consider the following model for a virus epidemic. Each node is either Susceptible to infection, or Infected, or Recovered from infection (the SIR model). At time 0 a single node is infected. At each timestep,

(i) Infected nodes may pass on the virus to susceptible nodes. The probability that an infected node infects a susceptible node is p, thus the probability that a susceptible node is not infected is $(1-p)^I$ where I is the number of infected nodes.
(ii) Infected nodes may recover. The probability that an infected node recovers is q.

Draw the state space diagram, for an epidemic in a system with N nodes. *[Hint. The size of the state space is much smaller than 3^N.]*

Let $N = 20$, $p = 1/3$, $q = 1/2$. Compute the probability that all nodes eventually become infected. Compute the expected time until the epidemic dies away. Investigate how these quantities depend on p and q. Repeat with $N = 200$.

Question 6. This question concerns the Gilbert model, which has been proposed as a model for noise in a wireless channel. The model specifies a Markov chain $X_n, n \geq 0$ where $X_n \in \{\text{good}, \text{bad}\}$ is the state of nature when the nth bit is sent. The transition probabilities are $P_{\text{goodbad}} = p$, $P_{\text{goodgood}} = 1-p$, $P_{\text{badgood}} = q$, and $P_{\text{badbad}} = 1-q$. In the good state, there are no errors. In the bad state, the bit may be corrupted with probability h.

(i) Let $E_n = 1$ if bit n is corrupted and 0 otherwise. Draw a state space diagram for the pair (X_n, E_n).
(ii) Calculate the equilibrium distribution of the Markov chain (X_n, E_n). What is the bit error rate, i.e. the long run fraction of bits that are corrupted?
(iii) The sequence $(E_n, n \geq 0)$ may be divided into runs of 0s and runs of 1s. For example, the sequence 00000111000110000 has two runs of 1s, of length 3 and 2. Calculate the distribution of the length of a run of 1s. What is the mean length of a run of errors?

You have derived two equations which relate the three unknown parameters (p, q and h) to two measurable statistics (bit error rate, mean length of a run of errors). One can also calculate the distribution of the length of a run of 0s, and this gives an extra equation; with three equations one can calculate all three unknown parameters.