Pigeonhole Principle

Also known as Dirichletdrawer principle, shoebox principle.
Principle: If $(n+1)$ objects are put into n-boxes, then atleast one box contains two or more of the objects.

1. Among 13 people there are atleast 2 who have their birthdays in the same month.

   13 - pigeons 12 - holes

2. Given m integers $a_1, a_2, ..., a_m$ there exists k and l with $1 \leq k < l \leq m$ such that $a_{k+1} + a_{k+2} + ... + a_l$ is divisible by m. Less formally, there exists consecutive a's in the sequence $a_1, a_2, ..., a_n$ whose sum is divisible by m. Solution: Consider the m - sums,

$$\begin{gathered} a_1 \\ a_1 + a_2 \\ a_1 + a_2 + a_3 \\ . \\ . \\ . \\ a_1 + a_2 + ... + a_n \end{gathered}$$

   If any of the sums is divisible by m, then the claim holds, Thus we may suppose that each of the sums has a non-zero remainder when divided by m, and so a remainder equal to one of $1, 2, ..., m-1$. Since these are 'm' sums and $(m-1)$ remainders, two of the sums have the same remainder when divided by m. Therefore there are integers k and l with $k < l$ such that $a_1 + a_2 + ... + a_k$ and $a_1 + a_2 + ... + a_l$ have the same remainder when divided by m.

$$\Rightarrow a_1 + a_2 + ... + a_k = bm + r \ , \ a_1 + a_2 + ...l = cm + r$$

Subtracting
$a_{k+1} + ... + a_l = (c-b)m$ and thus that $a_{k+1} + ... + a_l$ is divisible by m.

3. A chess master who has 11 weeks to prepare for a tournament decides to play atleast one game every day but in order not to tire himself he decides not to play more than 12 games during any one week. Show that there exists a succession of days during which he would have played exactly 21 games.

   <u>Solution:</u> Let a_i be the number of games played upto day i.
   Clearly $a_1 < a_2 < ... < a_{77}$. (Since atleast one game per day)
   Atmost 12 games per week $\Rightarrow a_{77} \leq 132$.
   Consider the sequence $a_1 + 21, a_2 + 21, ..., a_{77} + 21$.
   Clearly $a_1 + 21 < a_2 + 21 < ... < a_{77} + 21$
   Note that $a_1 < a_2 < ... < a_{77}$ and $a_1 + 21 < a_2 + 21 ... < a_{77} + 21$ together has 154 numbers.

   $$a_{77} < 132 \Rightarrow a_{77} + 21 \leq 153 \ .$$

   Thus each of the 154 numbers $a_1, a_2, ..., a_{77}, a1+21, ..., a_{77}+21$ is each an integer between 1 and 153. It follows that two of them are equal. Since $a1, ..., a_{77}$ and $a_1+21, ..., a_{77}+21$, it must be the case that they are distinct.

   There exists an 'i' and 'j' such that $a_i = a_j + 21$ therefore,
   days $j+1, j+2, ..., i$, the chess master played a total of 21 games.

4. From the integers $1, 2, ..., 200$, 101 are chosen. Show that among the integers chosen there are two such that one of them is divisible by the other.

   <u>Solution:</u> 100 holes: 1 3 199

   a number $i = r \times 2^k$, $k \geq 0$ r : odd. Put that i in the r^{th} box.

Suppose a box x contains

$$m = x.2^p$$
$$n = x.2^q,\ m, n \text{ are different} \Rightarrow p, q \text{ are different.}$$
$$m > n \Rightarrow p > q$$
$$\frac{m}{n} = 2^{p-q} \text{ hence the claim.}$$

Strong form:
Let $q_1, q_2, ..., q_n$ be positive integers. If $q_1 + q_2 + ... + q_n - n + 1$ objects are put into n-boxes, then either the first box contains at least q_1 objects or the second box q_2 objects ... or the n^{th} box contains atleast q_n objects.

Special case: $\forall i, q_i = r$.
If $n(r-1)+1$ objects are put into 'n' boxes, then atleast one of the boxes contains 'r' or more of the objects.

5. Two disks, one smaller than the other , are each divided into 200 congruent sectors. In the larger disk 100 of the sectors are chosen arbitrarily and painted red; the other 100 sectors are painted blue. In the smaller disk each sector is painted either red or blue with no stipulation on the number of red and blue sectors.The small disk is then placed on the larger disk so that their centers coincide.Show that it is possible to align the two disks so that the number of sectors of small disk whose color matches the corresponding sector of the large disk is atleast 100.

Solution: To see this we observe that if the larger disk is fixed in place, there are 200 possible positions for the smaller disk such that each sector of the small disk is contained in a sector of the large disk.We first count the total number of color matches over all of the 200 possible positions of the disks. Since the large disk has 100 sectors of each of the two colors, each sector of the smaller disk will match in color the correponding sector of the larger disk in exactly 100 of the 200 possible positions. Thus the total number of color matches over all the positions equals the number of sectors of the small disk multipiled by 100, and this equals 20,000. Therfore, the average number of color matches per position is $20,000/200 = 100$ .So there must be a position with atleast 100 color matches.

An alternative principle:
If n integers $m_1, m_2, ...m_n$ have an average $(m_1 + m_2 ... + m_n)/n$ which is greater than r-1 , then at least one of the integers $m_1, m_2, ..., m_n$ is greater than or equal to r.

Exercises:

1. Show that given any 52 intgers there exist two of them whose sum or else whose difference is divisible by 100.
2. Suppose that in a group of 6 persons each pair are either friends or enemies. Prove that among the 6, either there are 3 persons who are mutual friends or 3 persons who are mutual enemies. Show by example that the conclusion may be false for 5 persons.
3. Prove that in a group of n people there are two who have the same number of acquaintances in the group.
4. Prove that any 5 points chosen within a square of side length 2, there are two whose distance apart is atmost $sqrt(2)$.
5. A student has 31 days to prepare for an examination . From past experiance she knows that she will require no more than 60 hours of study. She also knows that she will wishes to study atleast 1 hour per day. Show that no matter how she schedules her study time (a whole number of hours per day however) there is a succession of days during which she will have studied exactly 13 hours.