CS603 Mathematical Concepts in Computer Science

Problem-Set-Vector Spaces

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1. In each of the following parts, a set V is given and some operations are defined. Check whether V is a vector space with these operations. If so, find the dimension and a basis. Justify your answers.
   (a.)
   $V = R^2$. For $(a_1, a_2), (b_1, b_2) \in V$ and $\alpha \in R$, define
   $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$
   $\alpha(a_1, a_2) = (0, 0)$ if $\alpha = 0$,
   $\alpha(a_1, a_2) = (\alpha a_1, \frac{a_2}{\alpha})$ if $\alpha \neq 0$
   (b.)
   $V = C^2$. For $(a_1, a_2), (b_1, b_2) \in V$ and $\alpha \in R$, define
   $(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 2b_2)$
   $\alpha(a_1, a_2) = (\alpha a_1, \alpha a_2)$
   (c).
   Let $V = R^2$, with the usual coordinatewise addition and coordinatewise scalar multiplication defined as $\alpha(a_1, a_2) = (a_1, 0)$.
   (d).
   $V = \{(x, y) \mid y \leq 0, x, y \in R\}$ with the usual coordinatewise addition and coordinatewise scalar multiplication.
   (e).
   $V = R^2$. Define $(a, b) + (c, d) = (ac, bd)$ and the usual coordinatewise addition and coordinatewise scalar multiplication.

2. In each of the following, a vector space V and a subset H is given. Check whether H is a subspace.
   (a).
   $V = R^2; H = \{(x_1, x_2) \mid x_2 = 2x_1 - 1\}$.
   (b.)
   $V = R^3; H = \{(x_1, x_2, x_3) \mid 2x_1 - x_2 - x_3 = 0\}$.
   (c).
   $V = R^3; H = \{(x, y, z) \mid xyz = 0\}$.
   (d).
   $V =$ the set of all 3×3 real matrices with the usual addition and scalar multiplication.
   - H=the set of all 3×3 matrices of trace zero.
   - H=the set of all 3×3 matrices with determinant zero.

3. In each of the following, determine whether the given set A of vectors in a vector space V is linearly dependent, and if it is, express one of the vectors as linear combinations of the remaining vectors.

   - $V = R^3; A = \{(1, 0, -1), (2, 5, 1), (0, -4, 3)\}$
   - $V = R^3; A = \{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$
   - V=set of all polynomials of degree 3; $A = \{x^2 - 3x + 5, x^3 + 2x^2 - x + 1, x^3 + 3x^2 - 1\}$.

4. Determine which of the following sets are bases for R^3.

   - $\{(1, 0, -1), (2, 5, 1), (0, -4, 3)\}$
   - $\{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$
   - $\{(1, -3, -2), (-3, 1, 3), (2, 5, 7)\}$.

Additional Questions: Strong Dose

1. Let $V = C^2, F = R$ Write a basis for this vector space.
2. Let $V = R^n$ and $F = R, W = \{\alpha = (a_1, \ldots, a_n) \mid a_1 a_2 = 0\}$. Is W a subspace of V?
3. Show that the vectors $(1, 0, -1), (1, 2, 1), (0, -3, 2)$ form a basis of R^3 over the field R.
4. Show that the set $\{1, 4 + 2x, 1 + 4x + 2x^2, 1 + x + 4x^2 + 2x^3\}$ is a basis for $P_3(x)$ which is the set of all polynomials with real coefficients and degree at most 3.
5. In R^4 with usual . and +. Let $U = span\{(1, 0, 0, 0), (0, 1, 0, 0)\}$ and let $W = span\{(1, 1, 0, 0)(0, 0, 1, 1)\}$. Find a basis for $U \cap W$.