MATHEMATICAL FOUNDATION OF COMPUTER SCIENCE
PROBLEM SHEET

1. Let the universe consist of all integers and let
   $P(x) : x$ is a prime,
   $Q(x) : x$ is positive,
   $E(x) : x$ is even,
   $N(x) : x$ is divisible by 9,
   $S(x) : x$ is a perfect square, and
   $G(x) : x$ is greater than 2.

   Then express each of the following in symbolic form :

   $a)$ x is even or x is a perfect square.
   $b)$ x is a prime and x is divisible by 9.
   $c)$ x is a prime and x is greater than 2.
   $d)$ If x is a prime, then x is greater than 2.
   $e)$ If x is a prime, then x is positive and not even.

2. Translate each of the following sentences into symbols, first using no existential quantifier, and second using no universal quantifier :

   $a)$ Not all cars have carburetors.
   $b)$ Some people are either religious or pious.
   $c)$ No dogs are intelligent.
   $d)$ All babies are illogical.
   $e)$ Every number is either negative or has a square root.
   $f)$ Some numbers are not real.
   $g)$ Every connected and circuit-free graph is a tree.
   $h)$ Not every graph is connected.

3. Prove or disprove the validity of the following arguments :

   $a)$ Every living thing is a plant or an animal.
   David's dog is alive and it is not a plant.
   All animals have hearts.
   Hence, David's dog has a heart.

   $b)$ No mathematicians are ignorant.
   All ignorant people are haughty.
   Hence, some haughty people are not mathematicians.

   $c)$ Babies are illogical.
   Nobody is despised who can manage a crocodile.
   Illogical people are despised.
   Hence, babies cannot manage crocodiles.

d) Students of average intelligence can do arithmetic.
A student without average intelligence is not a capable student.
Your students cannot do arithmetic.
Therefore, your students are not capable.

e) All integers are rational numbers.
Some integers are powers of 2.
Therefore, some rational numbers are powers of 2.

f) Some rational numbers are powers of 3.
All integers are rational numbers.
Therefore, some integers are powers of 3.

g) All clear explanations are satisfactory.
Some excuses are unsatisfactory.
Hence, some excuses are not clear explanations.

h) Some dogs are animals.
Some cats are animals.
Therefore, some dogs are cats.

i) All dogs are carnivorous.
Some animals are dogs.
Therefore, some animals are carnivorous.

4) The following propositions involve predicates that define sets. Use the properties to conclude relationships between these sets. Use Venn diagrams to check the validity of the arguments :

a) All cigarettes are hazardous to health.
All smokums are cigarettes.
Hence, all smokums are hazardous to health.

b) Some scientists are not engineers.
Some astronauts are not engineers.
Hence, some scientists are not astronauts.

c) All astronauts are scientists.
Some astronauts are engineers.
Hence, some engineers are scientists.

d) Some humans are vertebrates.
All humans are mammals.
Therefore, some mammals are vertebrates.

e) No mothers are males.
Some males are politicians.
Hence, some politicians are not mothers.

f) Some females are not mothers.
Some politicians are not females.
Hence, some politicians are not mothers.

g) All doctors are college graduates.
Some doctors are not golfers.
Hence, some golfers are not college graduates.

h) All fathers are males.
Some students are fathers.
Hence, all students are males.

i) All fathers are males.
Some students are fathers.
Hence, some students are males.

5) Use mathematical induction to prove that each of the following statements is true for all positive integers n :

a. If $a_n = 5a_{n-1} - 6a_{n-2}$ for $n >= 2$ and $a_0 = 12$ and $a_1 = 29$, then $a_n = 5(3^n) + 7(2^n)$.

b. $1^2 - 2^2 + 3^2 - 4^2 \ldots\ldots (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$

c. $x + y$ is a factor of the polynomial $x^{2n+1} + y^{2n+1}$.

d. $n(n^2 + 5)$ is an integer multiple of 6.

e. Show that the sum of the first n terms of an arithmetic progression with initial term a and common difference d is $\frac{n}{2}[2a + (n-1)d]$.

f. For each integer $n >= 4, n! > 2^n$.

g. For each integer $n >= 5, 2^n > n^2$.

h. Suppose that we have a system of currency that has \$3 and \$5 bills. Show that any debt of \$$n$ can be paid with only \$3 and \$5 bills for each integer $n >= 8$. Do the same problem for \$2 and \$7 bills and $n >= 9$.

i. Show that any integer composed of 3^n identical digits is divisible by 3^n. (For example, 222 *and* 555 *are divisible by* 3, *while* 222, 222, 222 *and* 555, 555, 555 *are divisible by* 9.)

j. For each integer $n >= 2$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} > \sqrt{n}$

6. (a) Show that
$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

(b) Show that
$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

(c) Show that
$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + ... + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$$

7. Show that for any integer n
$(11)^{n+2} + (12)^{2n+1}$ is divisible by 133.

8. A survey was conducted among 1000 people. Of these 595 are Democrats, 595 wear glasses, and 550 like ice cream; 395 of them are Democrats who wear glasses, 350 of them are Democrats who like ice cream, and 400 of them wear glasses and like ice cream; 250 of them are Democrats who wear glasses and like ice cream. How many of them who are not Democrats, do not wear glasses, and do not like ice cream ? How many of them are Democrats who do not wear glasses, and do not like ice cream?

9. It is known that at the university 60 percent of the professors play tennis, 50 percent of them play bridge, 70 percent jog, 20 percent play tennis and bridge, 30 percent play tennis and jog, and 40 percent play bridge and jog. If someone claimed that 20 percent of the professors jog and play bridge and tennis, would you believe this claim ? why ?

10. The 60,000 fans who attended the homecoming football game bought up all the paraphernalia for their cars. Altogether, 20,000 bumper stickers, 36,000 window decals, and 12,000 key rings were sold. We know that 52,000 fans bought at least one item and no one bought more than one of a given item. Also, 6000 fans bought both decals and key rings, 9000 bought decals and bumper stickers, and 5000 bought both key rings and bumper stickers.

a) How many fans bought all three items ?
b) How many fans bought exactly one item ?
c) Someone questioned the accuracy of the total number of purchasers; 52,000 (given that all the other numbers have been confirmed to be correct). This person claimed the total number of purchasers to be either 60,000 or 44,000. How do you dispel the claim ?

11. Out of a total of 130 students, 60 are wearing hats to class, 51 are wearing scarves, and 30 are wearing both hats and scarves. Of the 54 students who are wearing sweaters, 26 are wearing hats, 21 are wearing scarves, and 12 are wearing both hats and scarves. Everyone wearing neither a hat nor a scarf is wearing gloves.

    a) How many students are wearing gloves ?
    b) How many students not wearing a sweater are wearing hats but not scarves ?
    c) How many students not wearing a sweater are wearing neither a hat nor a scarf ?

12. Among 100 students, 32 study mathematics, 20 study physics, 45 study biology, 15 study mathematics and biology, 7 study mathematics and physics, 10 study physics and biology, and 30 do not study any of the three subjects.

    a) Find the number of students studying all three subjects.
    b) Find the number of students studying exactly one of the three subjects.

13. At a DAR(Daughters of American revolution) meeting of 30 women, 17 are descended from George Washington, 16 are descended from John Adams, and 5 are not descended from Washington or Adams. How many of the 30 women are descended from both Washington and Adams ?

14. Seventy-five children went to an amusement park where they can ride on merry-go-round, roller coaster, and ferris wheel. It is known that 20 of them have taken all the three rides, and 55 of them have taken at least two of the three rides. Each ride costs $0.50, and the total receipt of the amusement park was $70. Determine the number of children who did not try any of the rides.

15.(a) Among 50 students in a class, 26 got an A in the first examination and 21 got an A in the second examination. If 17 students did not get an A in either examination, how many students got an A in both examinations ?

(b) If the number of students who got an A in the first examination is equal to that in the second examination, if the total number of students who got an A in exactly one examination in 40, and 4 students did not get an A in either examination, determine the number of students who got an A in the first examination only, who got an A in the second examination only, and who got an A in both examinations.