MATHEMETICAL FOUNDATION OF COMPUTER SCIENCE PROBLEM SHEET 2 SOLUTIONS

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1. $A = \{1, 2\}$
   $\rho(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
   $\rho(A) \times A = \{(1, \emptyset), (2, \emptyset), (1, \{1\}), (2, \{1\}), (1, \{2\}), (2, \{2\}), (1, \{1, 2\}), (2, \{1, 2\})\}$

2. To show that $A \times B \subseteq C \times D$, consider $(a, b) \in A \times B$.
   Clearly $a \in A, b \in B$. Since $A \subseteq C, a \in C, B \subseteq D, b \in D$, this implies that $(a, b) \in C \times D$.
   Since we argued for an arbitrary pair $(a, b) \in A \times B$, it follows that $A \times B \subseteq C \times D$.

3. Given claim is wrong. Counter example:
   $A = \{1, 2\}, B = \emptyset$
   $C = \{3\}, D = \{4\}$
   $A \times B = \emptyset, C \times D = \{(3, 4)\}$
   Clearly $A \times B \subseteq C \times D$ but $A \nsubseteq C$

4. $A \times \emptyset = \emptyset$. The set is not well defined.

5. $A \times B = \emptyset$. At least one of the sets A or B is empty.

6. If $A = \emptyset$ then $A \subseteq A \times A$.

7.

qn	Reflexive	Symmetric	Anti-symmetric	Transitive	Equivalence	Poset
a)	$\times$	$\times$	$\surd$	$\times$	$\times$	$\times$
b)	$\times$	$\times$	$\surd$	$\times$	$\times$	$\times$
c)	$\times$	$\surd$	$\times$	$\surd$	$\times$	$\times$
d)	$\surd$	$\surd$	$\times$	$\surd$	$\surd$	$\times$

8. a. 2^{n^2-n}
   b. $2^{\frac{n(n+1)}{2}}$

c. $2^{\frac{n^2-n}{2}}$

d. $n!$

9. Given that $\forall a\, \exists b\, (b \in A \wedge (a,b) \in R)$. To prove R is reflexive;
since R is symmetric if $(a,b) \in R \Rightarrow (b,a) \in R$
and since R is transitive $(a,b) \in R, (b,a) \in R \Rightarrow (a,a) \in R$ and this argument is true $\forall a \in A$. Therefore R is reflexive. Hence R is an equivalence relation.

10. To prove that T is equivalence relation we need to prove T is reflexive, T is symmetric and T is transitive.
Given that $(a,b) \in T$ iff $(a,b),(b,a) \in R$
Clearly $(a,a) \in T\ \forall a \in A$, This is true because R is reflexive. This proves that T is reflexive.
If $(a,b) \in T$ we need to prove that $(b,a) \in T$. By the hypothesis (given condition), it is easy to see that $(b,a) \in T$. Hence T is symmetric.
If $(a,b) \in T$ and $(b,c) \in T$, we need to prove that $(a,c) \in T$.
$(a,b) \in T \rightarrow (a,b),(b,a) \in R$
$(b,c) \in T \rightarrow (b,c),(c,b) \in R$
Since R is transitive $(a,c) \in R$ and $(c,a) \in R$, this implies that $(a,c) \in T$. Hence T is transitive. Therefore, T is an equivalence relation.

11. Since R is reflexive $(a,a) \in R\ \forall a \in A$. Clearly $(a,a) \in S\ \forall a \in A$. This proves that S is reflexive.

To prove that S is symmetric,
$(a,b) \in S \rightarrow \exists x\ (a,x) \in R, (x,b) \in R$
Since R is symmetric $(x,a) \in R, (b,x) \in R$.
Therefore by given definition, $(b,a) \in S$.
This proves that S is symmetric.

To prove S is transitive,
If $(a,b) \in S$ and $(b,c) \in S$ we need to prove that $(a,c) \in S$.
$(a,b) \in S \rightarrow \exists d\ (a,d),(d,b) \in R$
R is symmetric $\rightarrow (d,a),(b,d) \in R$
$\Rightarrow (a,b) \in R, (b,a) \in R$
$(b,c) \in S \rightarrow\ \exists e\ (b,e),(e,c) \in R$
R is symmetric $\Rightarrow (e,b),(c,e) \in R$
$\Rightarrow (b,c) \in R, (c,b) \in R$
Since R is transitive, $(a,c) \in R, (c,a) \in R$ ——(1)
Since R is reflexive, $(c,c) \in R$ ——(2)

From (1) and (2) it follows that $(a,c) \in S$
Therefore, S is transitive and hence an equivalence relation.

12. Let R be a reflexive relation on a set A. Show that R is an equivalence relation iff (a, b) and (a, c) are in R implies that (b, c) is in R.

$\Rightarrow$: Given that R is an equivalence relation, we need to prove that $(a, b), (a, c) \in R \rightarrow (b, c) \in R$
Since R is symmetric, $(a, b) \in R \Rightarrow (b, a) \in R$
Since R is transitive, $(b, a), (a, c) \in R \Rightarrow (b, c) \in R$
Hence necessity is proved.

$\Leftarrow$: To show that R is an equivalence relation, we need to show that R is symmetric and transitive.
By definition, $(a, b), (a, c) \in R \Rightarrow (b, c) \in R$
Also $(a, c), (a, b) \in R \Rightarrow (c, b) \in R$
Therefore, R is symmetric.
To prove transitivity, if $(x, y), (y, z) \in R$ then $(x, z) \in R$
$(x, y) \in R,\ (a, x)\&(a, y) \in R$
$(y, z) \in R,\ (a, y)\&(a, z) \in R$
it follows that $(x, z) \in R$. Hence R is transitive.
Therefore R is an equivalence relation. Hence sufficiency is proved.

13. Let $A = \{1, 2, 3\}$

   a. Unary relation $= \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \emptyset\}$
   b. $|A \times A| = 3 \times 3 = 9$
   Number of binary relations: 2^9

14.
   a. Unary relation: Sets of all subsets of the set A.
   Number of subsets $= 2^n$
   Therefore, number of unary relations $= 2^n$
   b. Number of elements in the cross product $= n^2$
   Relation is a subset of a cross product
   Number of subsets $=$ number of relations $= 2^{n^2}$
   c. 2^{n^3}

15.

qn	Reflexive	Symmetric	Irreflexive	Anti-symmetric	Transitive
$R1$	×	×	×	×	√
$R2$	×	√	√	√	√

16. $A = \{1, 2\}$
$R = \{(1, 1), (1, 2)\}$ - neither reflexive nor irreflexive.
$A = \emptyset$ - Both reflexive and irreflexive.

17.
   a. To prove:$(R_2 \cup R_3)R_4 = R_2R_4 \cup R_3R_4$.
   Let $(b, d) \in (R_2 \cup R_3)R_4 \Leftrightarrow \exists c\ ((b, c) \in (R_1 \cup R_3) \wedge (c, d) \in R_4)$

$\Leftrightarrow \exists c\,(((b,c) \in R_2 \vee (b,c) \in R_3) \wedge (c,d) \in R_4)$
$\Leftrightarrow \exists c\,(((b,c) \in R_2 \wedge (c,d) \in R_4) \vee ((b,c) \in R_3 \wedge (c,d) \in R_4))$
$\Leftrightarrow \exists c\,((b,c) \in R_2 \wedge (c,d) \in R_4) \vee \exists c((b,c) \in R_3 \wedge (c,d) \in R_4)$
$\Leftrightarrow (b,d) \in R_2R_4 \vee (b,d) \in R_3R_4$
Therefore $(R_2 \cup R3)R_4 \Leftrightarrow R_2R_4 \cup R_3R_4$

b. To prove: $(R_2 \cap R3)R_4 \subset R_2R_4 \cap R_3R_4$.
Let $(b,d) \in (R_2 \cap R_3)R_4$
$\Leftrightarrow \exists c\,((b,c) \in R_2 \cap R_3 \wedge (c,d) \in R_4)$
$\Leftrightarrow \exists c\,(((b,c) \in R_2 \wedge (b,c) \in R_3) \wedge (c,d) \in R_4)$
$\Leftrightarrow \exists c\,(((b,c) \in R_2 \wedge (c,d) \in R_4) \wedge ((b,c) \in R_3 \wedge (c,d) \in R_4))$
$\Rightarrow \exists c\,((b,c) \in R_2 \wedge (c,d) \in R_4) \wedge \exists c\,((b,c) \in R_3 \wedge (c,d) \in R_4)$
$\Rightarrow ((b,d) \in R_2R_4) \wedge ((b,d) \in R_3R_4)$
Therefore,
$(R_2 \cap R_3)R_4 \;\subset\; R_2R_4 \cap R_3R_4$

18. a. Since R_1 and R_2 are reflexive on A, implies that
$\forall a \in A, (a,a) \in R_1$ and $(a,a) \in R_2$.
Clearly in $R_1R_2, \forall a \in A\ (a,a) \in R_1R_2$ hence R_1R_2 is reflexive.

b. $A = \{1,2\}, R_1 = \{(1,2)\}, R_2 = \{(2,1)\}$
R_1, R_2 are irreflexive. But $R_1R_2 = \{(1,1)\}$ which is NOT irreflexive.

c. $A = \{1,2,3\}, R_1 = \{(1,2),(2,1)\}, R_2 = \{(2,3),(3,2)\}$
R_1, R_2 are symmetric. $R_1R_2 = \{(1,3)\}$ is NOT symmetric.

d. $A = \{1,2,3\}, R_1 = \{(1,2),(3,2)\}, R_2 = \{(2,3),(2,1)\}$
R_1,R_2 are antisymmetric. $R_1R_2 = \{(1,3),(3,1),(3,3),(1,1)\}$ is NOT antisymmetric.

e. $A = \{1,2,3\}, R_1 = \{(1,2),(3,4)\}, R_2 = \{(2,3),(4,1)\}$
R_1,R_2 are transitive. $R_1R_2 = \{(1,3),(3,1)\}$ is NOT transitive.

19. a. Prove: $r(R_1 \cup R_2) = r(R_1) \cup r(R_2)$.
$R_1 \cup R_2 \supseteq R_1 \qquad R_1 \cup R_2 \supseteq R_2$
$r(R_1 \cup R_2) \supseteq r(R_1) \qquad r(R_1 \cup R_2) \supseteq r(R_2)$
$r(R_1 \cup R_2) \supseteq r(R_1) \cup r(R_2)$
Formally,
TPT: $r(R_1) \cup r(R_2) \subseteq r(R_1 \cup R_2)$
$r(R_1 \cup R_2)$ is a reflexive relation containing R_1. Since $r(R_1)$ is reflexive. By definition, $r(R_1) \subseteq r(R_1 \cup R_2)$. Similarly, $r(R_2) \subseteq r(R_1 \cup R_2)$. Therefore, $r(R_1) \cup r(R_2) \subseteq r(R_1 \cup R_2)$
Hence the claim. (Verify $r(R_1 \cup R_2) \subseteq r(R_1) \cup r(R_2)$)

b. Similar to above proof.

c. $(R_1 \cup R_2) \supset (R_1) \Rightarrow t(R_1 \cup R_2) \supset (R_1)$,
since $t(R_1 \cup R_2)$ is transitive by definition $t(R_1 \cup R_2) \supset t(R_1)$

Similarly, $t(R_1 \cup R_2) \supset t(R_2)$
Hence, $t(R_1 \cup R_2) \supset t(R_1) \cup t(R_2)$

d. $A = \{1, 2\}$
$R_1 = \{(1, 2)\} \quad R_2 = \{(2, 1)\}$
$R_1 \cup R_2 = \{(1, 2), (2, 1)\}$
$t(R_1) = \{(1, 2)\} \quad t(R_2) = \{(2, 1)\}$
$t(R_1 \cup R_2) = \{(1, 2), (2, 1), (1, 1)\}$
$t(R_1 \cup R_2) \neq t(R_1) \cup t(R_2)$.

20. $A = \{1, 2, ...n\}$
$R = \{(1, 2), (2, 3), (3, 4)...(n-1, n)\}$

21. a. R is a quasi order. We need to verify irreflexive, antisymmetric and transitive. It is enough to check irreflexive and transitive.
$\forall a \in A (a, a) \notin R$, clearly $(a, a) \notin R^c, \forall a \in A$
So R^c is irreflexive.
If $(a, b) \in R^c, (b, d) \in R^c$, to show that $(a, d) \in R^c$
$(a, b) \in R^c \Rightarrow (b, a) \in R$
$(b, d) \in R^c \Rightarrow (d, b) \in R$
since R is transitive, $(d, a) \in R \Rightarrow (a, d) \in R^c$
Hence R^c is a quasi order.

b. R is a partial order.
$\forall a \in A (a, a) \in R \Rightarrow (a, a) \in R^c$.
So R^c is reflexive.

For antisymmetry, prove: If $(a, b), (b, a) \in R^c$ then $a = b$
$(a, b) \in R^c \Rightarrow (b, a) \in R$
$(b, a) \in R^c \Rightarrow (a, b) \in R$
Since R is antisymmetric, $a = b$, thus R^c is antisymmetric.
If $(a, b) \in R^c, (b, c) \in R^c$, to show that $(a, c) \in R^c$
$(a, b) \in R^c \Rightarrow (b, a) \in R$
$(b, c) \in R^c \Rightarrow (b, c) \in R$
since R is transitive, $(c, a) \in R \Rightarrow (a, c) \in R^c$
Hence R^c is a partial order.

c. Given: R is a linear order. By above argument R^c is a poset. To prove that for every $a, b \in A$ either (a, b) or $(b, a) \in R^c$.
Since R is a linear order, clearly (a, b) or $(b, a) \in R$. Hence (b, a) or $(a, b) \in R^c$.
Hence the claim.

22. a. $A = I$
$R = <$
subset $= I^-$
Linear order but not well order.

b. Infinite:
$\sum = \{0,1\}$; $A = \sum^\star$; $R =$ containment ; subset $= \{\{0\}\{1\}\}$

c. Finite:
$R =$ Containment
subset $= \{\{a\}\{b\}\}$
glb $= \Phi$; No least element

d. $\{\{a\},\{b\}\}$ upper bound but no lub.

Infinite:
$\{0, \sqrt{2}\}$ Relation : less than defined over rational numbers in $\{0, \sqrt{2}\}$.
UB exists but no lub. UB is any number greater than $\sqrt{2}$.

23. a. Since R_1, R_2 are equivalence relations on A:
$\Rightarrow (a,a) \in R_1$ and R_2. $\forall a \in A \Rightarrow (a,a) \in R_1 \cap R_2$, $\forall a \in A$.
$R_1 \cap R_2$ is reflexive.
$(a,b) \in R_1 \cap R_2$. To show that $(b,a) \in R_1 \cap R_2$
$(a,b) \in R_1 \cap R_2 \Rightarrow (a,b) \in R_1$ and $(a,b) \in R_2$.
Since R_1 and R_2 are symmetric $(b,a) \in R_1$ and $(b,a) \in R_2$.
$\Rightarrow (b,a) \in R_1 \cap R_2$

$(a,b) \in R_1 \cap R_2$, $(b,c) \in R_1 \cap R_2$. To show that $(a,c) \in R_1 \cap R_2$.
$(a,b) \in R_1 \cap R_2 \Rightarrow (a,b) \in R_1 \cap (a,b) \in R_2$
$(b,c) \in R_1 \cap R_2 \Rightarrow (b,c) \in R_1 \cap (b,c) \in R_2$
Since R_1 and R_2 are transitive $(a,c) \in R_1$ and $(a,c) \in R_2$
$\therefore (a,c) \in R_1 \cap R_2$.
Therefore $R_1 \cap R_2$ is an equivalence relation on A.

b. $R_1 \cup R_2$ is not an equivalence relation on A.

24. Universal relation satisfies reflexive, transitive,symmetric trivially. Hence it is an equivalence relation.

25. The definition of reflexivity, symmetry and transitivity are vacously true on empty relation defined on Φ . Hence an equivalence relation.

Rank : 0

26. a. $\pi = \{(a,a)(b,b)(c,c)(a,b)(a,c)(b,a)(b,c)(c,a)(c,b)(d,d)\}$

b. $\pi_2 = \{(a,a),(b,b),(c,c),(d,d)\}$

$\pi_3 = \{(a,a)(a,b)(a,c)(a,d)(b,a)(b,b)(b,c)(b,d)(c,a)(c,b)(c,c)(c,d)(d,a)(d,b)(d,c)(d,d)\}$

c.

Rest of the questions are unimportant for Quiz-1

27. Let $A = I$. Define R_1, R_2, R_3 on A as follows:

$aR_1b \Leftrightarrow a \equiv b \bmod (3)$.

$aR_2b \Leftrightarrow a \equiv b \bmod (5)$.

$aR_3b \Leftrightarrow a \equiv b \bmod (6)$.

a. Draw a partial order diagram for the poset $< \{A|R_1, A|R_2, A|R_3,\}, refines >$

b. Describe the equivalence relations induced by
$(A|R_1).(A|R_3)$,
$(A|R_1) + (A|R_3)$,
$(A|R_1).(A|R_2)$,
$(A|R_1) + (A|R_2)$,
What are the rank of these relations?

28. Let $A = \phi$ and $B =$ any set. Is there a function from A to B? Is there a function from B to A?

29. Under what conditions is the length function which maps $\sum^\star$ to N a bijection?

30. Let A and B finite sets. Suppose $|A| = m$, $|B| = n$. State the relationship which must hold between $'m'$ and $'n'$ for each of the following to be true.

a. There exists an injection from A to B.

b. There exists an surjection from A to B.

c. There exists an bijection from A to B.

31. For each of the following sets A and B ,construct the bijection from A to B.

(1) $A = (0,1)$ $B = (0,2)$

(2) $A = I$ $B = N$

(3) $A = R$ $B = (0,\infty)$

(4) $A = \rho(\{a,b,c\})$ $B = 2^{\{a,b,c\}}$

(5) $A = [0,1)$ $B = (\frac{1}{4},\frac{1}{2}]$

32. Let f be a function from A to B, where A has $n \geq 2$ elements. State necessary conditions on B and for which the rank of the equivalence relation induced by $'f'$ on A is
(a) 1 (b) 2 (c) n.