INDIAN INSTITUTE OF TECHNOLOGY MADRAS

D

ADDITIONAL SHEET

ROLL NO :

5 a) $a_n = 5a_{n-1} - 6a_{n-2}$ $\quad a_n = 5(3^n) + 7(2^n)$

$\underline{n=0}$ $a_0 = 5(3^0) + 7(2^0) = 12$ ✓

Assume it is true for 'n'.

Consider,

$a_{n+1} = 5a_n - 6a_{n-1}$

$= 5[5(3^n) + 7(2^n)] - 6[5(3^{n-1}) + 7(2^{n-1})]$

$= (3+2)5 \cdot 3^n + (3+2)7 \cdot 2^n - 6[5(3^{n-1}) + 7(2^{n-1})]$

$= 5 \cdot 3^{n+1} + 7 \cdot 2^{n+1} + 10 \cdot 3^n + 21 \cdot 2^n - 10 \cdot 3^n - 21 \cdot 2^n$

$= 5 \cdot 3^{n+1} + 7 \cdot 2^{n+1}$ □.

5 b) $1^2 - 2^2 + 3^2 - 4^2 \dots = (-1)^{n-1} \frac{n(n+1)}{2}$

$n=1$ ✓

Hypothesis : True for n.

Indn: $1^2 - 2^2 + \dots (-1)^{n-1} n^2 + (-1)^n (n+1)^2$

$(-1)^{n-1} \frac{n(n+1)}{2} + (-1)^n (n+1)^2$

By hypothesis.

$= (n+1)\left[-\frac{n}{2} + (n+1)\right](-1)^n = (-1)^n \frac{(n+1)(n+2)}{2}$ □.

5 c) $x+y$ is a factor of the polynomial, $x^{2n+1} + y^{2n+1}$

$n=0$ $x+y$ ✓

Hypo true for 'n' ✓

Indn: We need to prove that $x^{2(n+1)+1} + y^{2(n+1)+1}$ is divisible by $x+y$.

Consider $(x^2+y^2)(x^{2n+1} + y^{2n+1})$, We want $x^2 \cdot x^{2n+1} + y^2 \cdot y^{2n+1}$

$\Rightarrow (x^2+y^2)(x^{2n+1} + y^{2n+1}) - x^2 \cdot y^{2n+1} - y^2 \cdot x^{2n+1} \underbrace{+ y^2 \cdot y^{2n+1} - y^2 \cdot y^{2n+1}}_{\text{Add \& subtract this term.}}$

$\Rightarrow (x^2+y^2)(x^{2n+1} + y^{2n+1}) + y^{2n+1}((x+y)(x-y)) - y^2(x^{2n+1} + y^{2n+1})$

Clearly the above expression is divisible by $x+y$. Hence the proof.

5 d) $n(n^2+5)$ is an integer multiple of 6.

$n=1$, $1(1+5)=6$ ✓

Hypo, true for n.

Indn. $(n+1)((n+1)^2+5) = (n+1)(n^2+1+2n+5)$

$= n(n^2+5) + n(2n+1) + n^2+5+2n+1$

$= n(n^2+5) + 2n^2+n+n^2+2n+6$

$= n(n^2+5) + 6\left(\frac{n^2}{2}+\frac{n}{2}+1\right)$ By the hypothesis $n(n^2+5)$ is an integer multiple of 6.

Clearly the above expression is an integer multiple of 6. □.

5 e) $a+(a+d)+(a+2d)+ \cdots +(a+(n-1)d) = \frac{n}{2}[2a+(n-1)d]$.

$n=1$ $\frac{1}{2}[2a+0] = a$ ✓

Hypo: Assume it is true for first 'n' terms.

Indn: Consider, $a+(a+d)+(a+2d)+\cdots+(a+(n-1)d)+a+nd$

$\frac{n}{2}[2a+(n-1)d]+a+nd$

$= \frac{n}{2}.2a + a + \frac{n}{2}(n-1)d + d$ $= \frac{2(n+1)}{2}a + d\left(\frac{n(n-1)+2n}{2}\right)$

$= \frac{n+1}{2}[2a+nd]$ □.

5 f. $n \geq 4$, $n! > 2^n$

Base: $n=4$ $4! > 2^4$ ✓

Hypo: $n! > 2^n$

Indn: $(n+1)! = n!(n+1) = n!n + n!$

$> 2^n + 2^n = 2^{n+1}$

therefore $\forall n \geq 4$ $n! > 2^n$.

5 g. $n \geq 5$, $2^n > n^2$.

Base: $2^5 > 5^2$ ✓

Hypo: $2^n > n^2$

Indn: $(n+1)^2 = n^2 + \underbrace{2n+1}$

$< 2^n + 2^n$

$< 2^{n+1}$.

$\therefore$ $\forall n \geq 5$ $2^n > n^2$.

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ROLL NO :

h) Base : $n=8$, It can be paid with \$3 & \$5 bills.

Hypo : Assume that debt of \$n can be paid with \$3 & \$5 bills.

Indn : Consider, debt of \$n+1.

This can be written as 3K or 3K+1 or 3K+2. If it is 3K+1, then it can be written as $3(K-3)+9+1$. Now we can use two \$5 bills and (K-3) \$3 bills. Indn is proved in this case.

If it is 3K+2. then it can be written as $3(K-1)+2+3$. Now we can use one \$5 bill and (K-1) \$3 bills. Indn is proved in this case.

If it is 3K, then \$3 bills are enough.

Similarly we can prove for \$2 and \$7 bills.

i)

Base : $n=1$ Any 3-digit integer with three identical digits is divisible by 3.

Hypo : True for n.

Indn : Let x be an integer composed of 3^{K+1} identical digits.

x can be written as $x = y \times z$

y : An integer composed of 3^K identical digits

$$z = 10^{2\cdot 3K} + 10^{3K} + 1 = \underbrace{100\cdots0}_{3^K-1\ 0's}\underbrace{0100\cdots0}_{3^K-1\ 0's}1$$

Since we assume that y is divisible by 3^K (by the hypothesis) and z is clearly divisible by 3, we conclude that x is divisible by 3^{K+1}.

j) For each integer $n \geq 2$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$

Base : $n = 2$ $\quad \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$

Hypo : Assume it is true for 'n'.

Indn : $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}$

$$> \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n} \cdot \sqrt{n+1} + 1}{\sqrt{n+1}} > \frac{\sqrt{n} \cdot \sqrt{n} + 1}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$$

therefore for all $n \geq 2$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$.

6 a) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Base : Easy (Verify) Hypo : Assume it is true for 'n'.

Indn : $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)}$

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2) + 1}{(n+1)(n+2)} = \frac{1}{(n+2)} \cdot \frac{(n^2 + 2n + 1)}{n+1} = \frac{(n+1)^2}{(n+1)(n+2)}$$

$$= \frac{(n+1)}{n+2}$$ $\therefore \forall n$ $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

□.

b) $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$ verify Base, Hypothesis.

Indn! $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} + \frac{1}{(2n+1)(2n+3)}$

$$= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} = \frac{1}{(2n+3)} \left[\frac{2n^2 + 3n + 1}{2n+1} \right] = \frac{(n+1)}{(2n+3)}.$$

$\therefore \forall n$ Given claim is true.

c) $\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$ verify Base, Hypothesis.

Indn: $\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{(3n-2)(3n+1)} + \frac{1}{(3n+1)(3n+4)}$

$$= \frac{n}{3n+1} + \frac{1}{(3n+1)(3n+4)} = \frac{1}{3n+4} \left[\frac{n(3n+4) + 1}{3n+1} \right] = \frac{1}{3n+4} \left[\frac{(3n+1)(n+1)}{3n+1} \right]$$

$$= \frac{(n+1)}{3n+4}.$$

$\therefore \forall n$ Given claim is true.

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ROLL NO :

7. S.T $\forall n \in I$ $(11)^{n+2} + (12)^{2n+1}$ is divisible by 133.

Base: $n=0$ $11^2 + 12^1 = 121+12 = 133$ divisible by 133.

Hypo: Assume it is true for 'n'.

Indn: $(11)^{n+1+2} + (12)^{2n+1+2}$

$= 11 \cdot 11^{n+2} + 12^2 \cdot 12^{2n+1} = 11 \cdot 11^{n+2} + (133+11) \cdot 12^{2n+1}$

$= 11 \left(11^{n+2} + 12^{2n+1}\right) + 133 \cdot 12^{2n+1}$

↳ By Hypothesis this is divisible by 133.

Therefore the above expression is divisible by 133.

$\therefore$ $\forall n \in I$ $(11)^{n+2} + (12)^{2n+1}$ is divisible by 133.

8) $N = 1000$ $D = 595$ $W = 595$ $I = 550$

$D \cap W = 395$ $D \cap I = 350$ $W \cap I = 400$ $D \cap W \cap I = 250$.

$\bar{D} \cap \bar{W} \cap \bar{I} = ?$ $D \cap \bar{W} \cap \bar{I} = ?$

1) $\bar{D} \cap \bar{W} \cap \bar{I} = \overline{D \cup W \cup I} = N - D \cup W \cup I$

$= 1000 - [595 + 595 + 550 - (395 + 350 + 400) + 250]$

2) $D \cap \bar{W} \cap \bar{I} = D \cap (\overline{W \cup I}) = D \cap (N - W \cup I) = D \cap N - D \cap (W \cup I)$

$= D \cap N - ((D \cap W) \cup (D \cap I)) = 595 - [395 + 350 - 250] = 100.$

9) $N = 100$ $T = 60$ $B = 50$ $J = 70$ $T \cap B = 20$ $T \cap B \cap J = 20$.

$T \cap J = 30$

$B \cap J = 40$

$100 = 60 + 50 + 70 - (20 + 30 + 40) + T \cap B \cap J$

$T \cap B \cap J = 10\%$. Given claim is wrong.

10) $N = 60,000$ $B = 20,000$ $W = 36,000$ $K = 12,000$

$W \cap K = 6000$ $W \cap B = 9000$ $B \cap K = 5000$.

$B \cup W \cup K = 52,000$.

a) $52,000 = 20,000 + 36,000 + 12,000 - (6000 + 9000 + 5000) + B \cap W \cap K$.

$\cancel{52000}$ $B \cap W \cap K = 4000$.

b) Exactly one :

$$B = \overset{20,000}{20K} - 9K - 5K + 4K = 10K$$
$$W = 36K - 6K - 9K + 4K = 25K$$
$$K = 12K - 6K - 5K + 4K = 5K.$$
$\Big\}$ 40,000.

11) $N = 130$ $M = 60$ $S = 51$ $H \cap S = 30$ $W = 54$ $H \cap W = 26$ $W \cap S = 21$ $W \cap H \cap S = 12$.

a) $\bar{H} \cap \bar{S} = \overline{H \cup S} = 130 - H \cup S = 130 - (60 + 51 - 30)$
$= 130 - 81 = 49$.

b) $\bar{W} \cap H \cap \bar{S} = H \cap \overline{W \cup S} = H \cap (N - W \cup S) = H - (H \cap W) \cup (H \cap S)$

$= 60 - [26 + 30 - 12] = 16$.

c) $\bar{W} \cap \bar{H} \cap \bar{S} = \overline{W \cup H \cup S} := 130 - [60 + 51 + 54 - (30 + 26 + 21) + 12]$
$= N - W \cup H \cup S = 30$.

12) $N = 100$ $M = 32$ $P = 20$ $B = 45$ $M \cap B = 15$ $M \cap P = 7$ $P \cap B = 10$

$\bar{M} \cap \bar{P} \cap \bar{B} = 30$

$\bar{M} \cap \bar{P} \cap \bar{B} = \overline{M \cup P \cup B} = N - M \cup P \cup B = 30$

$M \cup P \cup B = 70$.

$M \cup P \cup B = 32 + 20 + 45 - (15 + 7 + 10) + M \cap P \cap B$

$M \cap P \cap B = 5$.

b) Exactly one :
$M = 32 - 15 - 7 + 5 = 15$
$P = 20 - 7 - 10 + 5 = 8$
$B = 45 - 15 - 10 + 5 = \underline{25}$
48.

13) $N = 30$ $W = 17$ $A = 16$ $\bar{W} \cup \bar{A} = 5$ $W \cap A = ?$

$\bar{W} \cup \bar{A} = \bar{W} + \bar{A} - \bar{W} \cap \bar{A}$ $\bar{W} \cap \bar{A} = 13 + 14 - 5 = 22$

$\bar{W} \cap \bar{A} = \overline{W \cup A} = N - W \cap A$ $W \cap A = 30 - 22 = 8$.

ROLL NO :

14) $N = 75$ A, B, C are three events.

$A \cap B \cap C = 20$

20 students, 3-rides $\Rightarrow 20 \times 3 = 60$ rides $60 \times 0.5 = \$30$

Atleast two rides = 55 students.

$55 - 20 = 35$, 2-rides $\Rightarrow 35 \times 2 = 70$ rides $70 \times 0.5 = \$35$

total rides so far, $60 + 70 = 130$

total receipt $= \$70 \Rightarrow \frac{\$70}{0.5} = 140$ rides.

Remaining 10-rides taken by exactly 10 students (one student - one ride).

$10 \times 0.5 = \$5$

total no. of students participated in the ride $= 20 + 35 + 10 = 65$

No. of children who did not participate $= 10$.

15. a) $N = 50$ $F = 26$ $S = 21$ $\bar{F} \cap \bar{S} = 17$ $\overline{F \cup S} = 17$ $F \cup S = 33$

$F \cap S = F + S - F \cup S = 26 + 21 - 33 = 14$.

b) A : exam-I B : exam-II.

$A - (A \cap B) + B - (A \cap B) = 40$

$A + B - 2(A \cap B) = 40$

$\bar{A} \cap \bar{B} = 4$

$\overline{A \cup B} = 4$

$A \cup B = 46$

$A \cup B = A + B - A \cup B = 46$

$\underline{A + B - 2(A \cap B) = 40}$

$A \cap B = 6$

$A + B = 40$

$A = 20$ $B = 20$

Students who got an A in I-exam : 20

II-exam : 20

in Both : 6