MATHEMATICAL FOUNDATION OF COMPUTER SCIENCE
PROBLEM SHEET 2

1. Let $A = \{1, 2\}$. Construct the set $\rho(A) \times A$

2. Given that $A \subseteq C$ and $B \subseteq D$, show that $A \times B \subseteq C \times D$

3. Given that $A \times B \subseteq C \times D$, does it necessarily follow that $A \subseteq C$ and $B \subseteq D$.

4. Let A be an arbitrary set. Is the set $A \times \phi$ well defined?

5. Given that $A \times B = \phi$, what one can say about the set A and B?

6. Is it possible that $A \subseteq A \times A$ for some set A?

7. For each of the following check whether $'R'$ is Reflexive, Symmetric, Anti-symmetric, Transitive, An equivalence, A partial order.

   a. $R = \{(a, b) | a - b$ is an odd positive integers$\}$.
   b. $R = \{(a, b) | a = b^2$ where $a, b \in I^+\}$.
   c. Let P be the set of all people. Let R be a binary relation on P such that (a, b) is in R if a is a brother of b.
   d. Let R be a binary relation on the set of all strings of $0's$ and $1's$, such that $R = \{(a, b) | a \& b$ are strings that have same number of $0's$ $\}$.

8. Let A be a set with 10 distinct elements.
   How many different binary relations on A are there?

   a. How many of them are reflexive?
   b. How many of them are symmetric?
   c. How many of them are reflexive and symmetric?
   d. How many of them are total ordering relation?

9. Let R be a symmetric and transitive relation on set A. Show that if for every $'a'$ in A there exists $'b'$ in A, such that (a, b) is in R, then R is an equivalence relation.

10. Let R be a transitive and reflexive relation on A.Let T be a relation on A, such that (a, b) is in T iff both (a, b) and (b, a) are in R. Show that T is an equivalence relation.

11. Let R be a binary relation. Let $S = \{(a, b) | (a, c) \in R$ and $(c, b) \in R$ for some C $\}$. Show that if R is an equivalence relation, then S is also an equivalence relation.

12. Let R be a reflexive relation on a set A. Show that R is an equivalence relation iff (a, b) and (a, c) are in R implies that (b, c) is in R.

13. Let $A = \{1, 2, 3\}$

    a. List the unary relations on A.

    b. How many binary relations are there on A?

14. Let A be a set with n elements.

    a. Prove that there are 2^n unary relations on A.

    b. Prove that there are 2^{n^2} binary relations on A.

    c. How many ternary relations are there on A?

15. Consider the set of integers(I). For each of the following relations verify Reflexive, Symmetric, Irreflexive, Antisymmetric and Transitive.

    $R_1 =$"Divides with an integer quotient"(e.g $4D8$ but $4\neg D7$).

    $R_2 =$ Empty relation.

16. Find a non-empty set and a relation on it which is neither reflexive nor irreflexive. Choose the set to be as small as possible.What if the set is permitted to be empty?

17. Let R_1 be a relation from A to B, R_2 and R_3 be relations from B to C and R_4 be a relation from C to D . Then, prove that

    a. $(R_2 \cup R3)R_4 = R_2R_4 \cup R_3R_4$.

    b. $(R_2 \cap R3)R_4 \subset R_2R_4 \cap R_3R_4$.

18. Let R_1 and R_2 be arbitrary relations on A. Prove or Disprove the following assertions.

    a. if R_1 and R_2 are reflexive, then R_1R_2 is reflexive.

    b. if R_1 and R_2 are irreflexive, then R_1R_2 is irreflexive.

    c. if R_1 and R_2 are symmetric, then R_1R_2 is symmetric.

    d. if R_1 and R_2 are antisymmetric, then R_1R_2 is antisymmetric.

    e. if R_1 and R_2 are transitive, then R_1R_2 is transitive.

19. Let R_1 and R_2 be a relations on A. Prove each of the following.

    a. $r(R_1 \cup R2) = r(R_1) \cup r(R_2)$.

    b. $s(R_1 \cup R2) = s(R_1) \cup s(R_2)$

    c. $t(R_1 \cup R2) \supset t(R_1) \cup t(R_2)$

    d. Show by counter example that
    $t(R_1 \cup R_2) \neq t(R_1) \cup t(R_2)$.

20. Find a set A with n-elements and a relation R on A such that $R^1, R^2, ..., R^n$ are all distinct. This establishes the bound $t(R) = \cup_{i=1}^{n} R^i$.

21. Prove the following assertions:

    a. If R is a quasi order, then so is R^c.

    b. If R is a partial order, then so is R^c.

    c. If R is a linear order, then so is R^c.

22. Construct examples of the following sets.

    a. A non-empty linearly ordered set in which some subsets do not have a least element.

    b. A non-empty partially ordered set which is not linearly ordered and in which some subsets do not have a greatest element. Construct both finite and infinite examples.

    c. A partially ordered set with a subset for which there exists a *glb* but which does not have a least element. Construct both finite and infinite examples.

    d. A partially ordered set with a subset for which there exists an upper bound but not a least upper bound. Construct both finite and infinite examples.

23. Let R_1 and R_2 be equivalence relations on a set A. Then $R_1 \cap R_2$ is an equivalence relation. Is $R_1 \cup R_2$ an equivalence relation?

24. Prove that the universal relation on any set A is an equivalence relation.

    What is the rank of this relation?

25. Prove that the empty relation is an equivalence relation on ϕ.

    What is the rank of this relation?

26. Suppose $A = \{a, b, c, d\}$ and π_1 is the following partition of A:

$$\pi_1 = \{\{a, b, c\}, \{d\}\}$$

    a. List the ordered pairs of the equivalence relation induced by π_1.

    b. Do the same for the partitions
    $\pi_2 = \{\{a\}, \{b\}, \{c\}, \{d\}\}$
    $\pi_3 = \{\{a, b, c, d\}\}$

    c. Draw a poset diagram of the poset $< \{\pi_1, \pi_2, \pi_3\}, refines >$.

27. Let $A = I$. Define R_1, R_2, R_3 on A as follows:
$aR_1b \Leftrightarrow a \equiv b \bmod (3)$.
$aR_2b \Leftrightarrow a \equiv b \bmod (5)$.
$aR_3b \Leftrightarrow a \equiv b \bmod (6)$.

a. Draw a partial order diagram for the poset
$< \{A|R_1, A|R_2, A|R_3,\}, refines >$

b. Describe the equivalence relations induced by
$(A|R_1).(A|R_3)$,
$(A|R_1) + (A|R_3)$,
$(A|R_1).(A|R_2)$,
$(A|R_1) + (A|R_2)$,
What are the rank of these relations?

28. Let $A = \phi$ and $B =$ any set. Is there a function from A to B? Is there a function from B to A?

29. Under what conditions is the length function which maps $\sum^\star$ to N a bijection?

30. Let A and B finite sets. Suppose $|A| = m$, $|B| = n$. State the relationship which must hold between $'m'$ and $'n'$ for each of the following to be true.

a. There exists an injection from A to B.

b. There exists an surjection from A to B.

c. There exists an bijection from A to B.

31. For each of the following sets A and B ,construct the bijection from A to B.

(1) $A = (0, 1)$ $B = (0, 2)$

(2) $A = I$ $B = N$

(3) $A = R$ $B = (0, \infty)$

(4) $A = \rho(\{a, b, c\})$ $B = 2^{\{a,b,c\}}$

(5) $A = [0, 1)$ $B = (\frac{1}{4}, \frac{1}{2}]$

32. Let f be a function from A to B, where A has $n \geq 2$ elements. State necessary conditions on B and for which the rank of the equivalence relation induced by $'f'$ on A is
(a) 1 (b) 2 (c) n.